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Convergence theorem of the Mann iteration for a class of generalized monotone nonexpansive mappings

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Abstract

In this paper, we concerned with convergence of the Mann iteration for finding an order fixed point of a monotone mapping that satisfies condition (Cλ) in an ordered Banach space. As consequence, several convergence theorems for monotone nonexpansive mappings are deduced. Our results not only include the recent ones announced by Dehaish and Khamis [8] and Song et al. [10] as special cases but also are established the weaker assumptions.

Keywords: convergence, Mann iteration, monotone mapping, nonexpansive mapping, ordered Banach space.

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1 Introduction

Throughout this paper, X denotes a Banach space and N is the set of natural numbers. Let C be a subset of X and T : C → C be a mapping. A point z ∈ X is said to be a fixed point of T if Tz = z. The set of fixed points of T is denoted by F(T). The mapping T : C → C is said to be nonexpansive if ∥Tx − Ty∥ ≤ ∥x − y∥ for all x, y ∈ C.

The convergence theorem of nonexpansive mappings have been considered by many researchers in recent years. In 1953, Mann [3] introduced the following iteration to find a fixed point of a nonexpansive mapping T, which referred as the Mann iteration,

\[ x_{n+1} = \beta_n T(x_n) + (1 - \beta_n)x_n \quad (1.1) \]

for each n ≥ 1 and x_1 ∈ C where \{\beta_n\} is a sequence in [0, 1].

In 2008, Suzuki [6] introduced some condition on mappings, condition (C), that weaker than nonexpansiveness and obtained the convergence theorems for mappings that satisfy such condition. In 2011, Garcia et al. [7] defined a kind of generalization of condition (C) as condition (C_λ).

On the other hand, fixed point theory in partially ordered metric spaces has been initiated by Ran and Reurings [5] for finding applications to metrix equation. Recently, Dehaish and...
Khamsi [8] introduced the concept of a monotone nonexpansive mapping in a Banach space $X$ endowed with the partial order $\preceq$. Let $C$ be a subset of $X$ and a mapping $T : C \to C$ is called

1. monotone if $Tx \leq Ty$ for all $x, y \in C$ with $x \preceq y$;
2. monotone nonexpansive if $T$ is monotone and

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$ with $x \preceq y$.

They used the Mann iteration for finding some order fixed points of monotone nonexpansive mappings in $X$ and obtained the weak convergence of Mann iteration that provided $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$. Very recently, Song et al. [10] proved some convergence theorems of the Mann iteration for finding an order fixed point of monotone nonexpansive mapping in a uniformly convex Banach space under the condition $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$.

In [9], monotone ($C_1$)–conditions are defined in $L_1[a, b]$. They presented existence of fixed point of such mapping.

In this paper, we mainly consider the Mann iteration for monotone mappings that satisfy condition ($C_1$) in an ordered Banach space $X$ with the partial order $\preceq$. In Section 3, we prove some weak and strong convergence theorems of the Mann iteration for monotone mappings that satisfy condition ($C_1$) in a uniformly convex Banach space which are extended and improved [10]. Moreover, convergence results without uniformly convexity are presented.

## 2 Preliminaries

Throughout this paper, let $X$ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\preceq$. Let $\to$ and $\rightharpoonup$ denote as the strong and weak convergences, respectively. In this paper, we assume that the order intervals are closed and convex.

A space $X$ is said to be uniformly convex if, for all $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x + y\|}{2} < 1 - \delta$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$.

**Definition 2.1.** Let $C$ be a nonempty subset of $X$ and $T : C \to C$ be a monotone mapping. For $\lambda \in [0, 1)$, $T$ is said to satisfy condition ($C_\lambda$) if for all $x, y \in C$ with $x \preceq y$ and $\lambda \|x - Tx\| \leq \|x - y\|$ implies

$$\|Tx - Ty\| \leq \|x - y\|. \quad (2.1)$$

**Remark 2.2.**

1. If $0 \leq \lambda_1 < \lambda_2 < 1$, then the condition ($C_{\lambda_1}$) implies condition ($C_{\lambda_2}$) but the converse fails. For example, let $T : [0, 1] \to [0, 1]$ defined by

$$Tx = \begin{cases} 
\frac{x}{2} & x \neq 1, \\
1 + \frac{\lambda}{2 + \lambda} & x = 1 
\end{cases}$$

where $\lambda \in (0, 1)$. Then the mapping $T$ satisfies condition ($C_{\lambda}$) but it fails condition ($C_{\lambda'}$) whenever $0 \leq \lambda' < \lambda$ (see [7, Example 5]).

2. $T$ is monotone nonexpansive if and only if a monotone mapping $T$ satisfies condition ($C_0$).

The next example is a direct generalization of a monotone nonexpansive mapping.
Example 2.3 ([9], Example 1). Let $C = \{f \in L_p[0, 3] : f(x) = a\}$, where $a \in [0, 3]$.
For $f, g \in C$, consider the partial relation $f \preceq g$ iff $f(x) \preceq g(x) \forall x \in [0, 3]$. Let $T : C \to C$ be defined by $T(f) = \begin{cases} 1, & f = 3, \\ 0, & f \neq 3. \end{cases}$

Then $T$ satisfies condition $(C_1)$.

Lemma 2.4. Let $C$ be a nonempty subset of $X$ and $T : C \to C$ be a monotone mapping. Assume that $T$ satisfies condition $(C_\lambda)$. Then, for $x, y \in C$ with $x \preceq y$, the following hold:

1. $\|Tx - T^2x\| \leq \|x - Tx\|$.
2. Either $\lambda\|x - Tx\| \leq \|x - y\|$ or $(1 - \lambda)\|Tx - T^2x\| \leq \|Tx - y\|$.
3. Either $\|Tx - Ty\| \leq \|x - y\|$ or $\|T^2x - Ty\| \leq \|Tx - y\|$.

Proof. It follows from [6, Lemma 5].

Lemma 2.5. Let $C$ be nonempty subset of $X$. Assume that a monotone mapping $T : C \to C$ satisfies condition $(C_\lambda)$. Then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ for all $x, y \in C$ with $x \preceq y$.

Proof. It follows from [6, Lemma 7].

Remark 2.6. If a monotone mapping $T$ satisfies condition $(C_\lambda)$, then $T$ is monotone quasi-nonexpansive, i.e., $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$ and $z \in F(T)$ with $x \preceq z$.

Lemma 2.7 ([4], Lemma 2). Let $\{z_n\}$ and $\{w_n\}$ be two sequences of $X$ and $\{\beta_n\}$ be a sequence in $[0, 1]$. Suppose that

$$z_{n+1} = (1 - \beta_n)z_n + \beta_n w_n \text{ for all } n \geq 1.$$ 

If the following properties are satisfied:

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n \to \infty} \beta_n < 1$;
(ii) $\lim_{n \to \infty} \|z_n\| = d$ and $\limsup_{n \to \infty} \|w_n\| \leq d$;
(iii) $\left\{\sum_{i=1}^{n} \beta_i w_i\right\}$ is bounded.

Then $d = 0$.

3 Main Results

From now on, we consider the Mann iteration process as follows: let $x_1 \in C$

$$x_{n+1} = \beta_n T(x_n) + (1 - \beta_n)x_n, \quad n \in \mathbb{N}$$

(3.1)

where $\{\beta_n\}$ is a sequence in $[0, 1]$ and we denote

$F_\preceq(T) = \{p \in F(T) : p \preceq x_1\}$ and $F_\succeq(T) = \{p \in F(T) : x_1 \preceq p\}$.

The following lemma is showed by Dehaish and Khamsi [8].

Lemma 3.1 ([8], Lemma 3.1). Let $C$ be a nonempty subset of $X$. Let $T : C \to C$ be a monotone mapping. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \preceq Tx_1$.

(i) $x_n \preceq x_{n+1} \preceq Tx_n \preceq Tx_{n+1}$;
(ii) $x_n \preceq x$ for all $n \in \mathbb{N}$ provided $\{x_n\}$ weakly converges to a point $x \in C$. 

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Lemma 3.3. Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T : C \to C \)
be a monotone quasi-nonexpansive mapping. Assume that the sequence \( \{x_n\} \) defined by (3.1) 
with \( x_1 \leq T x_1 \) and \( F_\geq(T) \neq \emptyset \). Then

(i) the sequence \( \{x_n\} \) is bounded;

(ii) \( \lim_{n \to \infty} \|x_n - z\| \) and \( \lim_{n \to \infty} d(x_n, F_\geq(T)) \) exist provided \( d(x_n, F_\geq(T)) \) denotes the distance from \( x \) to \( F_\geq(T) \).

Proof. Let \( z \in F_\geq(T) \). Since \( x_1 \leq z \) and \( T \) is monotone, \( T x_1 \leq z \). By Lemma 3.1 (i), we get
\[
x_n \leq x_{n+1} \leq T x_n \leq z.
\]
Since \( T \) is monotone quasi-nonexpansive, we get
\[
\|x_{n+1} - z\| = \|(1 - \beta_n)(x_n - z) + \beta_n(T x_n - z)\|
\leq (1 - \beta_n)\|x_n - z\| + \beta_n\|T x_n - z\|
\leq (1 - \beta_n)\|x_n - z\| + \beta_n\|x_n - z\|
= \|x_n - z\|.
\]
(3.2)
Then \( \{\|x_n - z\|\} \) is nonincreasing and bounded. Hence (i) and (ii) follow. This completes the proof. \( \square \)

3.1 Convergence results in uniformly convex Banach spaces

In this subsection, we present some convergence theorems for monotone mappings \( T \) that satisfies condition \((C_\alpha)\) using the Mann iteration process (3.1). In the sequel, we also need the following lemma.

Lemma 3.2 (\cite{2}, Theorem 2). Let \( r > 0 \) be a fixed real number. If \( X \) is a uniformly convex Banach space, then there exists a continuous strictly increasing convex function \( g : [0, +\infty) \to [0, +\infty) \) with \( g(0) = 0 \) such that
\[
\|\eta x + (1 - \eta)y\|^2 \leq \eta\|x\|^2 + (1 - \eta)\|y\|^2 - \eta(1 - \eta)g(\|x - y\|)
\]
for all \( x, y \in B_r(0) := \{u \in X : \|u\| \leq r\} \) and \( \eta \in [0, 1] \).

Lemma 3.4. Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) and \( T : C \to C \) be a monotone quasi-nonexpansive mapping. Assume that the sequence \( \{x_n\} \) defined by (3.1) with \( x_1 \leq T x_1 \) and \( F_\geq(T) \neq \emptyset \). If \( \sum_{n=1}^\infty \beta_n(1 - \beta_n) = \infty \), then
\[
\liminf_{n \to \infty} \|T x_n - x_n\| = 0.
\]
Proof. Let \( z \in F_\geq(T) \). By Lemma 3.2, we have \( \{\|x_n - z\|\} \) and \( \{\|T x_n - z\|\} \) are bounded. We may assume that such sequences belong to \( B_r(0) \) where \( r > 0 \). By Lemma 3.3, we get
\[
\|x_{n+1} - z\|^2 = \|(1 - \beta_n)(x_n - z) + \beta_n(T x_n - z)\|^2
\leq (1 - \beta_n)\|x_n - z\|^2 + \beta_n\|T x_n - z\|^2 - \beta_n(1 - \beta_n)g(\|x_n - T x_n\|)
\leq \|x_n - z\|^2 - \beta_n(1 - \beta_n)g(\|x_n - T x_n\|)
\]
and so
\[
\beta_n(1 - \beta_n)g(\|x_n - T x_n\|) \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\]
(3.3)
Therefore, we get
\[
\sum_{n=1}^\infty \beta_n(1 - \beta_n)g(\|x_n - T x_n\|) < \infty.
\]
Since $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$ and $g$ is continuous strictly increasing,
\[
\liminf_{n \to \infty} \|x_n - T x_n\| = 0.
\]
This completes the proof. \hfill \Box

Recall that $X$ is said to satisfy Opial property if there exists sequence $\{x_n\}$ in $X$ such that $x_n \rightharpoonup x$. Then,
\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|
\]
for all $y \in X$ with $x \neq y$.

Next, we present weak convergence theorem as follows.

**Theorem 3.5.** Let $X$ be a uniformly convex Banach space with the Opial property. Let $C$ be a nonempty closed convex subset of $X$ and a monotone mapping $T : C \to C$ satisfy condition $(C_\lambda)$. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \leq T x_1$ and $F_\lambda(T) \neq \emptyset$.

If $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$ and $\beta_n \geq \lambda$, then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Proof.** Notice that
\[
\lambda\|x_n - T x_n\| = \frac{\lambda}{\beta_n}\|x_{n+1} - x_n\| \leq \|x_{n+1} - x_n\|.
\]
Since $T$ satisfies condition $(C_\lambda),
\[
\|T x_n - T x_{n+1}\| \leq \|x_n - x_{n+1}\|.
\]
Then
\[
\|x_{n+1} - T x_{n+1}\| \leq (1 - \beta_n)\|x_n - T x_n\| + \|T x_n - T x_{n+1}\|
\]
\[
\leq (1 - \beta_n)\|x_n - T x_n\| + \|x_n - x_{n+1}\|
\]
\[
= \|x_n - T x_n\|.
\]
It means that
\[
\lim_{n \to \infty} \|x_n - T x_n\| \exists.
\]
It follows from Lemmas 3.2 (i) and 3.4 that the sequence $\{x_n\}$ is bounded and
\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0.
\]

By Lemma 3.1(i), there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in C$. By Lemma 3.1(ii), $x_1 \leq x_{n_j} \leq z$. It follows from Lemma 2.5 that
\[
\|x_{n_j} - T z\| \leq 3\|x_{n_j} - T x_{n_j}\| + \|x_{n_j} - z\|.
\]
This implies that
\[
\liminf_{j \to \infty} \|x_{n_j} - T z\| \leq \liminf_{j \to \infty} \|x_{n_j} - z\|.
\]
Suppose that $T z \neq z$. By the Opial property, we have $z \in F_\lambda(T)$.

Next we will show that $x_n \rightharpoonup z$. Suppose that $\{x_n\}$ has two subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ such that $x_{n_j} \rightharpoonup z, x_{n_k} \rightharpoonup w$. By similar argument as for $z \in F_\lambda(T)$, we have $w \in F_\lambda(T)$.

By Lemma 3.2(ii), $\lim_{n \to \infty} \|x_n - z\|$ and $\lim_{n \to \infty} \|x_n - w\|$ exist. By the Opial property, we have
\[
\lim_{n \to \infty} \|x_n - z\| = \lim_{j \to \infty} \|x_{n_j} - z\| < \lim_{j \to \infty} \|x_{n_j} - w\|
\]
\[
= \lim_{n \to \infty} \|x_n - w\| = \lim_{k \to \infty} \|x_{n_k} - w\|
\]
\[
\leq \lim_{k \to \infty} \|x_{n_k} - z\| = \lim_{n \to \infty} \|x_n - z\|
\]

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\[ \lim_{n \to \infty} \|x_n - z\| < \lim_{n \to \infty} \|x_n - z\|. \]

which is contraction. Thus \( \{x_n\} \) converges weakly to \( z \in F_\infty(T) \). □

If \( \lambda = 0 \) in Theorem 3.5, then \( T \) is monotone nonexpansive. The weak convergence results are presented as follow corollary.

**Corollary 3.6** ([10], Theorem 3.10). Let \( X \) be a uniformly convex Banach space with the Opial property. Let \( C \) be a nonempty closed convex subset of \( X \) and a mapping \( T : C \to C \) be monotone nonexpansive. Assume that the sequence \( \{x_n\} \) defined by (3.1) with \( x_1 \in Tx_1 \) and \( F_\infty(T) \neq \emptyset \). If \( \sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty \), then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

Recall that a mapping \( T : C \to C \) satisfy condition (I) [1] with respect to \( K \) if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(r) > 0 \) for all \( r \in (0, \infty) \) such that \( f(d(x, K)) \leq \|x - Tx\| \) for all \( x \in C \), where \( d(x, K) \) denotes the distance of \( x \) from \( K \).

Next, we present the strong convergence of \( T \) by assume that \( T \) satisfies condition (I).

**Theorem 3.7.** Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) and a monotone mapping \( T : C \to C \) satisfy condition (C\( \lambda \)). Assume that the sequence \( \{x_n\} \) defined by (3.1) with \( x_1 \in Tx_1 \) and \( F_\infty(T) \neq \emptyset \). If \( T \) satisfies condition (I) with respect to \( F_\infty(T) \) and \( \sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty \), then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Proof.** Since \( T \) satisfies condition (I) with respect to \( F_\infty(T) \),

\[ f(d(x_n, F_\infty(T))) \leq \|x_n - Tx_n\|, \]

where \( f : [0, \infty) \to [0, \infty) \) is nondecreasing function with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, \infty) \). It follows from Lemmas 3.2(ii) and 3.4, that

\[ \lim_{n \to \infty} d(x_n, F_\infty(T)) = 0. \]

Let \( \{x_{n_j}\} \) be a subsequence \( \{x_n\} \) and \( \{z_j\} \) be a sequence in \( F_\infty(T) \) such that \( \|x_{n_j} - z_j\| \leq \frac{1}{2^j} \) for all \( j \geq 1 \). Then, by (3.2),

\[ \|x_{n_{j+1}} - z_j\| \leq \|x_{n_j} - z_j\| \leq \frac{1}{2^j} \]

(3.5)

and so

\[ \|z_{j+1} - z_j\| \leq \|z_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - z_j\| \]

\[ \leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \]

\[ \leq \frac{1}{2^{j+1}}. \]

Then \( \{z_j\} \) is Cauchy in \( F_\infty(T) \). Since \( F_\infty(T) \) is closed, \( \{z_j\} \) converges to some \( z \in F_\infty(T) \). Moreover,

\[ \|x_{n_j} - z\| \leq \|x_{n_j} - z_j\| + \|z_j - z\| \]

\[ \leq \frac{1}{2^j} + \|z_j - z\| \to 0 \text{ as } j \to \infty. \]

Since \( \lim_{n \to \infty} \|x_n - z\| \exists \), we get

\[ \lim_{n \to \infty} \|x_n - z\| = 0. \]

It means that \( \{x_n\} \) converges strongly to \( z \) of \( T \). This completes the proof. □

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If $\lambda = 0$ in Theorem 3.7, then $T$ is monotone nonexpansive. The strong convergence results are presented as follow corollary.

**Corollary 3.8.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and a mapping $T : C \to C$ be monotone nonexpansive. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \leq Tx_1$ and $F_\geq(T) \neq \emptyset$. If $T$ satisfies condition (I) with respect to $F_\geq(T)$ and $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$, then $\{x_n\}$ converges strongly to a fixed point of $T$.

### 3.2 Convergence results without uniformly convexity

In this section, we consider the convergence theorem of $T$ that satisfies condition $(C_\lambda)$ in general Banach space.

**Theorem 3.9.** Let $C$ be a closed convex subset of $X$. Let a monotone mapping $T : C \to C$ satisfy condition $(C_\lambda)$. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \leq Tx_1$ and $F_\geq(T) \neq \emptyset$. If $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lambda \leq \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, then

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$

**Proof.** We will apply Lemma 2.7. From the iteration (3.1), we have

$$z_{n+1} = (1 - \beta_n)z_n + \beta_n w_n$$

where $z_n = x_n - Tx_n$ and $w_n = \beta_n^{-1}(Tx_n - Tx_{n+1})$. As the proof of Lemma 3.4,

$$\lim_{n \to \infty} \|z_n\| = \lim_{n \to \infty} \|x_n - Tx_n\| := d$$

Moreover,

$$\limsup_{n \to \infty} \|w_n\| = \limsup_{n \to \infty} \|\beta_n^{-1}(Tx_n - Tx_{n+1})\|$$

$$\leq \limsup_{n \to \infty} \beta_n^{-1} \|x_n - x_{n+1}\|$$

$$= \limsup_{n \to \infty} \|x_n - Tx_n\| = d$$

Finally, let $z \in F_\geq(T)$. By Remark 2.6 and $x_n \leq z$ for all $n \geq 1$, we have

$$\left\| \sum_{i=1}^{n} \beta_i w_i \right\| = \left\| \sum_{i=1}^{n} \beta_i \beta_i^{-1}(Tx_i - Tx_{i+1}) \right\|$$

$$= \|Tx_1 - Tx_{n+1}\|$$

$$\leq \|Tx_1 - z\| + \|z - Tx_{n+1}\|$$

$$\leq \|x_1 - z\| + \|z - x_{n+1}\|.$$

By Lemma 3.2, we have that $\left\{ \sum_{i=1}^{n} \beta_i w_i \right\}$ is bounded. It follows then that $d = 0$. This completes the proof. \qed

We now have the following convergence theorems without uniformly convexity.

**Theorem 3.10.** Let $X$ be a reflexive Banach space. Let $C$ be a nonempty closed convex subset of $X$. Let a monotone mapping $T : C \to C$ satisfy condition $(C_\lambda)$. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \leq Tx_1$ and $F_\geq(T) \neq \emptyset$. If $X$ satisfies Opial property, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lambda \leq \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, then $\{x_n\}$ converges weakly to a fixed point of $T$. 
Theorem 3.11. Let $C$ be a closed convex subset of $X$. Let a monotone mapping $T : C \to C$ satisfy condition $(C_{\lambda})$. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \preceq Tx_1$ and $F_\preceq(T) \neq \emptyset$. If $T$ satisfies condition (I) with respect to $F_\preceq(T)$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lambda \leq \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, then $\{x_n\}$ converges strongly to a fixed point of $T$.

If $\lambda = 0$ in Theorem 3.10 and Theorem 3.11, then the convergence results are presented as follows.

Corollary 3.12. Let $X$ be a reflexive Banach space. Let $C$ be a nonempty closed convex subset of $X$. Let $T : C \to C$ be monotone nonexpansive. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \preceq Tx_1$ and $F_\preceq(T) \neq \emptyset$. If $X$ satisfies Opial property, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n \to \infty} \beta_n < 1$, then $\{x_n\}$ converges weakly to a fixed point of $T$.

Corollary 3.13. Let $C$ be a closed convex subset of $X$. Let $T : C \to C$ be monotone nonexpansive. Assume that the sequence $\{x_n\}$ defined by (3.1) with $x_1 \preceq Tx_1$ and $F_\preceq(T) \neq \emptyset$. If $T$ satisfies condition (I) with respect to $F_\preceq(T)$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n \to \infty} \beta_n < 1$, then $\{x_n\}$ converges strongly to a fixed point of $T$.

Remark 3.14. By using the same ideas and techniques, we can also discuss the convergence results for a monotone mapping $T$ that satisfies condition $(C_{\lambda})$ with $F_\preceq(T) \neq \emptyset$.

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